2021 NREL Workshop on
Resilient Autonomous Energy Systems

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Approximating Feasible Power Injection Regions of Radial AC Networks via Dual SOCP

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ArXiv: 2109.02294. Funded by Hong Kong Research Grants Council (ECS No. 24210220)

Feasible power injection regions in AC networks

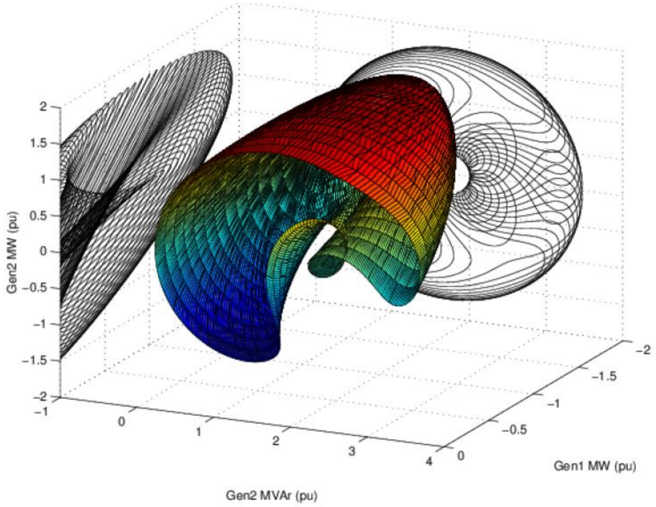
Nonlinear, nonconvex, implicit (intertwined with voltages, currents, line power flows, etc.)

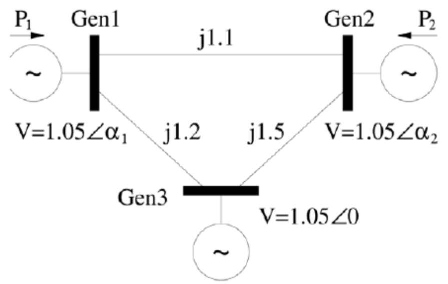
A simple closed-form approximation is important for resilient grid applications, e.g.,

- Solve OPF quickly for fast-timescale control in grid restoration
- Decide hosting capacities of renewable energy sources

“Feasible”: the power injections and their associated voltages, currents, etc. satisfy:

- Physical laws of circuit **“solvability”**
- Operational limits **“safety”**





I. A. Hiskens, R. J. Davy, 2001

Prior efforts

A review:

Molzahn, Hiskens, 2017

[Chapter 4.5]

DC approximation (convex polyhedral regions)

- J. Zhao, T. Zheng, E. Litvinov, 2014.
- W. Wei, F. Liu, S. Mei, 2014.

- Simple computation
- DC is coarse for **distribution networks**

AC *solvability* proved by fixed point theorems

- S. Bolognani, S. Zampieri, 2015. **(Banach)**
- C. Wang, A. Bernstein, J.-Y. Le Boudec, M. Paolone, 2016. **(Banach)**
- K. Dvijotham, H. Nguyen, K. Turitsyn, 2017. **(Brouwer)**
- J. W. Simpson-Porco, 2017. **(Brouwer)**

- Accurate AC models, reliable results
- How to incorporate *safety limits*?

Convex optimization for inner approximations

- M. Nick, R. Cherkaoui, J.-Y. Le Boudec, M. Paolone, 2017. **(Tightened-relaxed SOC)**
- H. D. Nguyen, K. Dvijotham, K. Turitsyn, 2018. **(Linear bounds for nonlinear terms)**
- N. Nazir, M. Almassalkhi, 2019. **(Constant estimates for nonlinear terms)**

- Both solvability and safety are addressed
- Explore feasible region in a specific shape/direction of the power-injection vector

This work

A closed-form polyhedral approximation of feasible power injection regions in radial AC networks

- Simple form and moderate computation
- Built through *dual* second-order cone program (SOCP), a convex program that preserves nonlinearity of AC power flow
- Fulfills both solvability and safety
- No need to specify a shape/direction of power-injection vector

Problem statement

AC dist-flow equations for a radial network (Solvability): [Baran, Wu, 1989]

$$\forall i \to j: \quad P_{ij} - r_{ij}\ell_{ij} - \sum_{k:j\to k} P_{jk} + p_j = 0 \quad (1a)$$

$$Q_{ij} - x_{ij}\ell_{ij} - \sum_{k:j\to k} Q_{jk} + q_j = 0 \quad (1b)$$

$$v_i - v_j - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r_{ij}^2 + x_{ij}^2)\ell_{ij} = 0 \quad (1c)$$

$$P_{ij}^2 + Q_{ij}^2 - v_i\ell_{ij} = 0. \quad (1d)$$

Active/reactive power injections: $(p, q) \in \mathbb{R}^{2N}$

State $x := (v, \ell, P, Q) \in \mathbb{R}^{4N}$

N = number of lines

= number of nodes excluding root/slack node

Safety limits:

$$\underline{v}_i \le v_i \le \overline{v}_i, \quad \forall i = 1, \ldots, N \quad (2a)$$

$$0 \le \ell_{ij} \le \overline{\ell}_{ij}, \quad \forall i \to j \quad (2b)$$

…… nodal voltage magnitudes

…… on-line current magnitudes

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$$P_{ij}^2 + Q_{ij}^2 - v_i\ell_{ij} = 0. \quad \text{(1d)}$$

Safety limits:

$$\underline{v}_i \le v_i \le \overline{v}_i, \quad \forall i = 1, \dots, N \quad \text{(2a)}$$

$$0 \le \ell_{ij} \le \overline{\ell}_{ij}, \quad \forall i \to j \quad \text{(2b)}$$

Let $(p, q) = (d, u) \in \mathbb{R}^{2N}$, $2N = D + U$

- Known constant injections: $d \in \mathbb{R}^D$
- Unknown variable injections: $u \in \mathbb{R}^U$

A power-injection vector u is ***feasible*** if there exists $x = (v, \ell, P, Q)$ such that $(x; d, u) = (x; p, q)$ satisfies (1)(2).

The ***feasible power injection region*** is

$$\mathcal{U} := \{u \in \mathbb{R}^U \mid u \textit{ is feasible.}\}$$

Our goal: find a closed-form approximation of $\mathcal{U}$

Problem statement

AC dist-flow equations for a radial network (Solvability): [Baran, Wu, 1989]

$$
\begin{aligned}
\forall i \to j: \quad & P_{ij} - r_{ij}\ell_{ij} - \sum_{k:j\to k} P_{jk} + p_j = 0 && \text{(1a)}\\
& Q_{ij} - x_{ij}\ell_{ij} - \sum_{k:j\to k} Q_{jk} + q_j = 0 && \text{(1b)}\\
& v_i - v_j - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r_{ij}^2 + x_{ij}^2)\ell_{ij} = 0 && \text{(1c)}\\
& P_{ij}^2 + Q_{ij}^2 - v_i\ell_{ij} = 0. && \text{(1d)}
\end{aligned}
$$

Safety limits:

$$
\begin{aligned}
& \underline{v}_i \le v_i \le \overline{v}_i, \quad \forall i = 1, \ldots, N && \text{(2a)}\\
& 0 \le \ell_{ij} \le \overline{\ell}_{ij}, \quad \forall i \to j && \text{(2b)}
\end{aligned}
$$

Feasibility problem for u:

Slack variables

$$
\begin{aligned}
\mathrm{FP}(u): \quad & \min \ 1^\intercal \tilde{z}\\
& \text{over } x = (v, \ell, P, Q), \ \tilde{z} = (z_s, z_q, \tilde{z}_q) \ge 0\\
& \text{s. t. } A_f x + B_f u + \gamma_f = 0\\
& A_s x + \gamma_s \le z_s\\
& P_{ij}^2 + Q_{ij}^2 - v_i \ell_{ij} \le z_{q,ij}, \ \forall i \to j\\
& v_i \ell_{ij} - (P_{ij}^2 + Q_{ij}^2) \le \tilde{z}_{q,ij}, \ \forall i \to j
\end{aligned}
$$

An equivalent definition of the ***feasible power injection region:***

$$\mathcal{U} = \{u \in \mathbb{R}^U \mid \mathrm{fp}(u) = 0\}$$

where $\mathrm{fp}(u)$ is the min. obj. val. of $\mathrm{FP}(u)$

Step 1: Convex relaxation of feasibility problem

Feasibility problem for u:

Slack variables

$$\begin{aligned} \text{FP}(u): \quad & \min\ 1^\intercal \tilde{z} \\ & \text{over } x = (v, \ell, P, Q),\ \tilde{z} = (z_s, z_q, \tilde{z}_q) \geq 0 \\ & \text{s. t. } A_f x + B_f u + \gamma_f = 0 \\ & \quad A_s x + \gamma_s \leq z_s \\ & \quad P_{ij}^2 + Q_{ij}^2 - v_i \ell_{ij} \leq z_{q,ij}, \ \forall i \to j \\ & \quad v_i \ell_{ij} - (P_{ij}^2 + Q_{ij}^2) \leq \tilde{z}_{q,ij}, \ \forall i \to j \end{aligned}$$

An equivalent definition of the ***feasible power injection region:***

$$\mathcal{U} = \{u \in \mathbb{R}^U \mid \text{fp}(u) = 0\}$$

where $\text{fp}(u)$ is the min. obj. val. of $\text{FP}(u)$

SOCP relaxation:

Slack variables

$$\begin{aligned} \text{FP}'(u): \quad & \min\ 1^\intercal z \\ & \text{over } x,\ y,\ z = (z_s, z_q) \geq 0 \\ & \text{s. t. } A_f x + B_f u + \gamma_f = 0 \\ & \quad A_s x + \gamma_s \leq z_s \\ & \quad y = A_y x + b_y \\ & \quad \|y_{ij}\|_2 \leq c_{q,ij} x + \gamma_{q,ij} + z_{q,ij}, \ \forall i \to j \end{aligned}$$

SOCP-relaxed feasible region:

$$\mathcal{U}' := \{u \in \mathbb{R}^U \mid \text{fp}'(u) = 0\}$$

where $\text{fp}'(u)$ is the min. obj. val. of $\text{FP}'(u)$

Step 2: Dual SOCP

SOCP relaxation:

Slack variables

$$\mathrm{FP}'(u): \quad \min\ 1^\intercal z \quad \text{over } x,\ y,\ z = (z_s, z_q) \geq 0$$
$$\text{s. t. } A_f x + B_f u + \gamma_f = 0$$
$$A_s x + \gamma_s \leq z_s$$
$$y = A_y x + b_y$$
$$\|y_{ij}\|_2 \leq c_{q,ij} x + \gamma_{q,ij} + z_{q,ij}, \ \forall i \to j$$

SOCP-relaxed feasible region:

$$\mathcal{U}' := \{u \in \mathbb{R}^U \mid \mathrm{fp}'(u) = 0\}$$

where $\mathrm{fp}'(u)$ is the min. obj. val. of $\mathrm{FP}'(u)$

Dual SOCP:

$$\mathrm{DP}'(u): \quad \max_{\mu,\lambda} \quad \mu_f^\intercal (B_f u + \gamma_f) + \lambda_s^\intercal \gamma_s - \mu_y^\intercal b_y - \lambda_q^\intercal \gamma_q$$
$$\text{s. t.} \quad 0 \leq \lambda \leq 1 \tag{6a}$$
$$A_f^\intercal \mu_f + A_s^\intercal \lambda_s = A_y^\intercal \mu_y + c_q^\intercal \lambda_q \tag{6b}$$
$$\|\mu_{y,ij}\|_2 \leq \lambda_{q,ij}, \quad \forall i \to j \tag{6c}$$

Slater's condition, i.e., (strict) feasibility holds for $\mathrm{FP}'(u)$ ⟹ *Strong duality*

An equivalent definition of the ***SOCP-relaxed*** feasible region:

$$\mathcal{U}' = \{u \in \mathbb{R}^U \mid \mathrm{dp}'(u) = 0\}$$
$$= \{u \in \mathbb{R}^U \mid D_u(\mu, \lambda) \leq 0,\ \forall (\mu, \lambda) \text{ satisfying } (6)\}$$

where $\mathrm{dp}'(u)$ is the max. obj. val. of $\mathrm{DP}'(u)$.

$\mathcal{U}'$ *is convex.*

Step 3: Closed-form approximation of $\mathcal{U}'$

Algorithm 1: Approximate relaxed feasible region $\mathcal{U}'$

1 **Initialization:** $\mathcal{U}'_{poly} = \{u \in \mathbb{R}^U \mid \underline{u} \le u \le \overline{u}\}$ for sufficiently low $\underline{u}$ and high $\overline{u}$; $\mathcal{V}_{safe} = \emptyset$; $c = 0$;

2 update vertices $\mathcal{V}\left(\mathcal{U}'_{poly}\right)$. Let $\mathrm{dp}'_{max} = 0$;

3 **for** $u \in \mathcal{V}\left(\mathcal{U}'_{poly}\right)$ *and* $u \notin \mathcal{V}_{safe}$ **do**

4   solve $\mathrm{DP}'(u)$ to obtain an optimal solution (μ^*, λ^*) and maximum objective value $\mathrm{dp}'(u)$;

5   **if** $\mathrm{dp}'(u) > \mathrm{dp}'_{max}$ **then**

6     $\mathrm{dp}'_{max} \leftarrow \mathrm{dp}'(u)$;

7     $(\mu_{max}, \lambda_{\max}) \leftarrow (\mu^*, \lambda^*)$;

8   **else if** $\mathrm{dp}'(u) \le 0$ **then** $\mathcal{V}_{safe} = \mathcal{V}_{safe} \cup \{u\}$;

9 **end**

10 **if** $\mathrm{dp}'_{max} = 0$ *or* $c = C_{max}$ **then**

11   return $\mathcal{U}'_{poly}$.

12 **else**

13   add to $\mathcal{U}'_{poly}$ a cutting plane: $\mu^{\intercal}_{f,max}(B_f u + \gamma_f) + \lambda^{\intercal}_{s,max}\gamma_s \le \mu^{\intercal}_{y,max} b_y + \lambda^{\intercal}_{q,max}\gamma_q$;

14   $c \leftarrow c + 1$;

15   go back to Line 2;

16 **end**

$$\mathcal{U}' = \{u \in \mathbb{R}^U \mid \mathrm{dp}'(u) = 0\}$$
$$= \{u \in \mathbb{R}^U \mid D_u(\mu, \lambda) \le 0,\ \forall(\mu, \lambda) \text{ satisfying (6)}\}$$

Main idea of Alg. 1:

- Start with an over-estimate convex polyhedron;
- Traverse its vertices; for each vertex u, solve $\mathrm{DP}'(u)$; if $\mathrm{dp}'(u) \le 0$, then $u \in \mathcal{U}'$ and is not checked again.
- Record the vertex u with highest $\mathrm{dp}'(u) > 0$, i.e., violating $\mathcal{U}'$ the most; add a cutting plane to remove this u; update polyhedron and vertices;
- Terminate Alg. 1 if $\mathrm{dp}'(u) \le 0$ for all vertices u (or if maximum number of iterations is reached); otherwise Repeat.

Step 3: Closed-form approximation of $\mathcal{U}'$

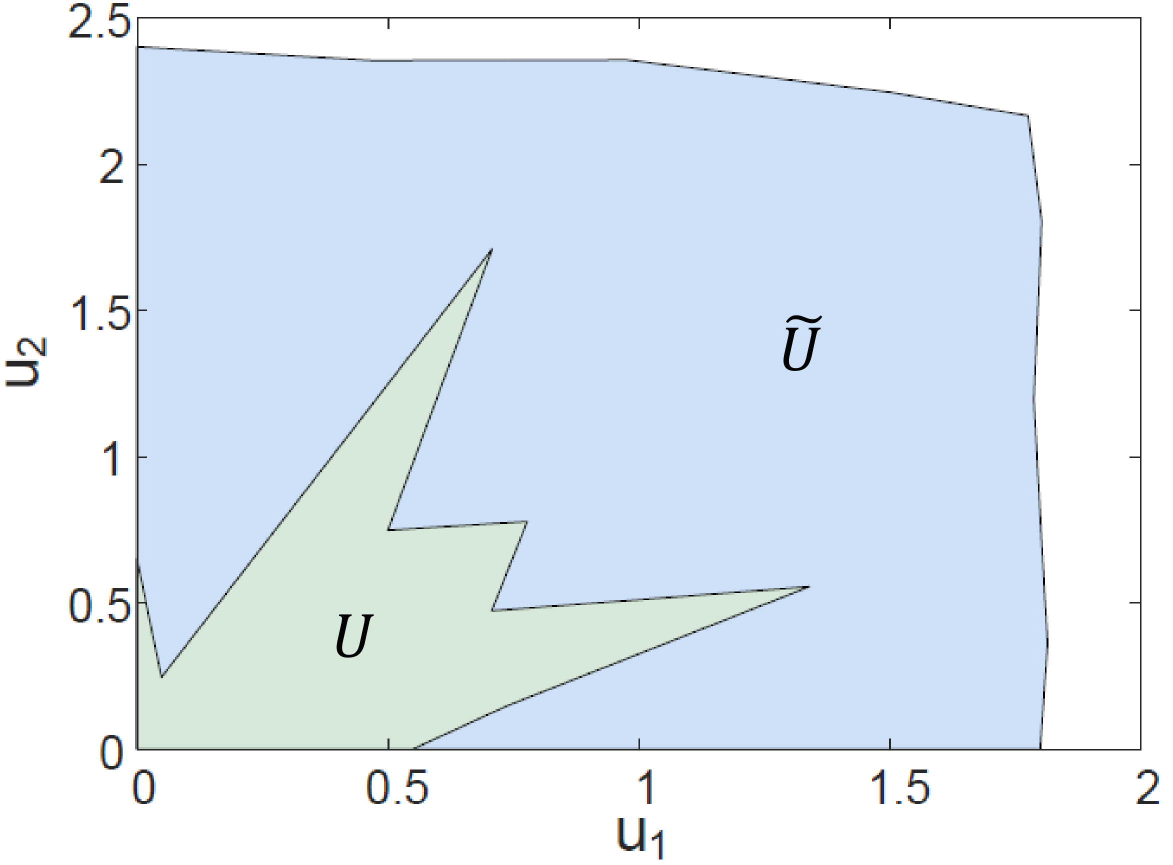
Proposition 3. *The output $\mathcal{U}'_{poly}$ in an arbitrary iteration of Algorithm 1 is an outer approximation of $\mathcal{U}'$.*

$\mathrm{dp}'(u) = 0$, i.e., $u \in U'$, for all vertices u

Proposition 4. *If Algorithm 1 terminates with $\mathrm{dp}'_{max} = 0$ in a finite number of iterations, then it returns a convex polyhedron $\mathcal{U}'_{poly} = \mathcal{U}'$.*

Corollary: *If $\mathcal{U}'$ is **not** a polyhedron, then Algorithm 1 **cannot** terminate in a finite number of iterations with $dp'_{max} = 0$.*

Step 4: Removing **SOCP-inexact** injections from $\mathcal{U}'$



$$\mathcal{U} = \mathcal{U}' \backslash \tilde{\mathcal{U}}$$

Proposition 5. *For every $u' \in \mathcal{U}'$, there must be $u \leq u'$ (element-wise), such that $u \in \mathcal{U}$.*

Consistent with the "load over-satisfaction" condition for exact SOCP relaxation of OPF

Definition 2. *A power injection $u \in \mathcal{U}'$ is* ***SOCP-inexact****, if every optimal solution of* $\mathrm{FP}'(u)$ *satisfies:*

$$\|y_{ij}\|_2 < c_{q,ij}x + \gamma_{q,ij} \quad \textit{for some } i \to j.$$

The ***SOCP-inexact power injection region*** *is:*

$$\tilde{\mathcal{U}} = \{u \in \mathcal{U}' \mid u \textit{ is SOCP-inexact}\}.$$

Step 4: Removing **SOCP-inexact** injections from $\mathcal{U}'$

Definition 2. *A power injection $u \in \mathcal{U}'$ is **SOCP-inexact**, if every optimal solution of* $\mathrm{FP}'(u)$ *satisfies:*

$$\|y_{ij}\|_2 < c_{q,ij}x + \gamma_{q,ij} \quad \textit{for some } i \to j.$$

*The **SOCP-inexact power injection region** is:*

$$\tilde{\mathcal{U}} = \{u \in \mathcal{U}' \mid u \textit{ is SOCP-inexact}\}.$$

An approximate definition from Dual SOCP:

$$\tilde{\mathcal{U}}_d := \{u \in \mathcal{U}' \mid \text{Every optimal solution of } \mathrm{DP}'(u) \text{ satisfies } \lambda_{q,ij} = 0 \text{ for some } i \to j\}.$$

Due to complementary slackness, $\tilde{\mathcal{U}} \subseteq \tilde{\mathcal{U}}_d$.

Focus on $\tilde{\mathcal{U}}_d$ as a good approximation of $\tilde{\mathcal{U}}$.

Step 4: Removing **SOCP-inexact** injections from $\mathcal{U}'$

Algorithm 2: Approximate $\tilde{\mathcal{U}}_d$ (or SOCP-inexact $\tilde{\mathcal{U}}$)

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1 **Initialization:** $\tilde{\mathcal{U}}_{poly} = \mathcal{U}'_{poly}$ returned by Algorithm 1.
  Given $\delta \in \mathbb{R}^N_+$, $\eta, \eta' \in \mathbb{R}_+$; $\mathcal{V}_{safe} = \emptyset$; $c = 0$;
2 update vertices $\mathcal{V}\left(\tilde{\mathcal{U}}_{poly}\right)$. Let $\text{dp}''_{max} = -\eta$;
3 **for** $u \in \mathcal{V}\left(\tilde{\mathcal{U}}_{poly}\right)$ *and* $u \notin \mathcal{V}_{safe}$ **do**
4   solve $\text{DP}''(u, \delta)$ to obtain an optimal solution (μ^*, λ^*) and maximum objective value $\text{dp}''(u, \delta)$;
5   **if** $\text{dp}''(u, \delta) > \text{dp}''_{max}$ **then**
6     $\text{dp}''_{max} \leftarrow \text{dp}''(u, \delta)$;
7     $(\mu_{max}, \lambda_{\max}) \leftarrow (\mu^*, \lambda^*)$;
8   **else if** $\text{dp}''(u, \delta) \le -\eta$ **then** $\mathcal{V}_{safe} = \mathcal{V}_{safe} \cup \{u\}$;
9 **end**
10 **if** $\text{dp}''_{max} = -\eta$ *or* $c = C_{max}$ **then**
11   return $\tilde{\mathcal{U}}_{poly}$.
12 **else**
13   add to $\tilde{\mathcal{U}}_{poly}$ a cutting plane:
   $\mu^{\intercal}_{f,max}(B_f u + \gamma_f) + \lambda^{\intercal}_{s,max}\gamma_s \le$
   $\mu^{\intercal}_{y,max} b_y + \lambda^{\intercal}_{q,max}\gamma_q - \eta'$;
14   $c \leftarrow c + 1$;
15   go back to Line 2;
16 **end**

$$\tilde{\mathcal{U}}_d := \{u \in \mathcal{U}' \mid \text{Every optimal solution of } \text{DP}'(u) \text{ satisfies } \lambda_{q,ij} = 0 \text{ for some } i \to j\}.$$

Tighten the dual feasible set to exclude $\lambda_q = 0$:

$$\lambda_q \ge \delta$$

For $u \in \tilde{\mathcal{U}}_d$, the *tightened* Dual SOCP $\text{DP}''(u, \delta)$ should attain max. value ***strictly*** lower than 0.

In Alg. 2, this requirement is represented by

$$\text{dp}''(u, \delta) \le -\eta < 0$$

Step 4: Removing **SOCP-inexact** injections from $\mathcal{U}'$

Alg. 2 returns a convex polyhedral approximation of $\widetilde{U}_d$ (or $\widetilde{U}$);

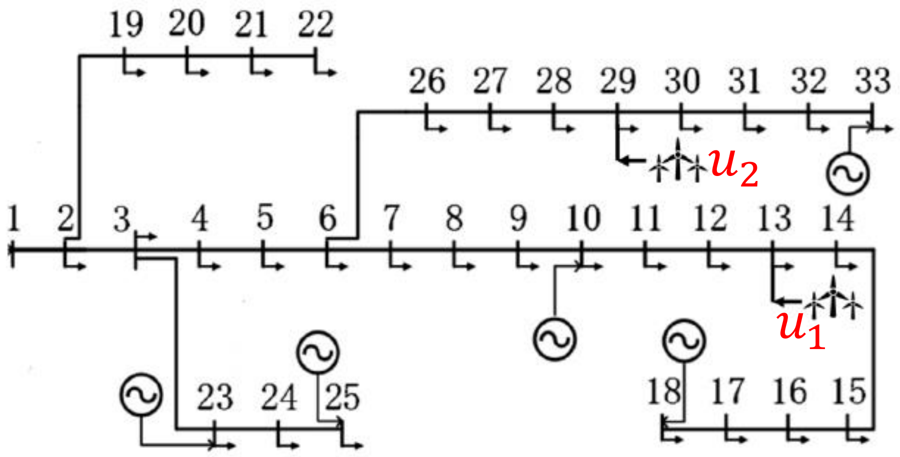
However, $\widetilde{U}$ is generally nonconvex;

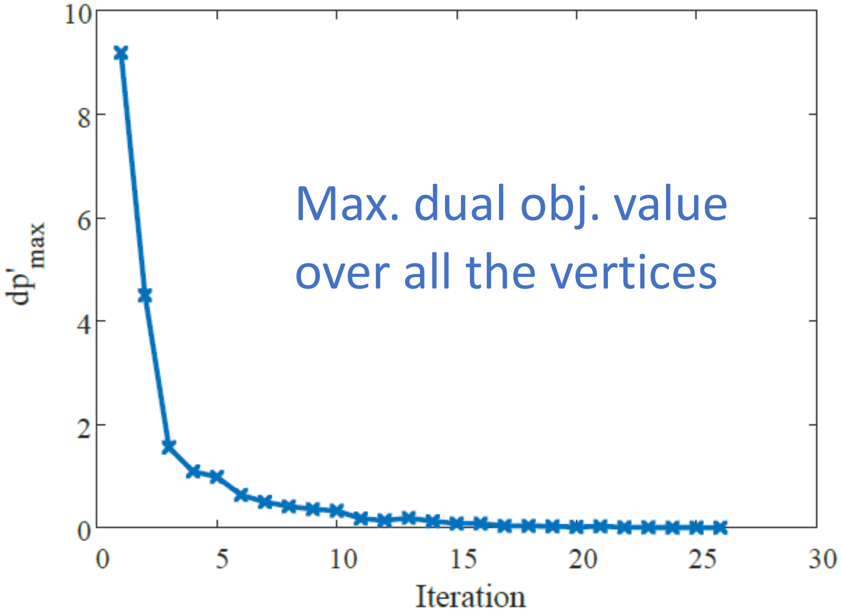
Moreover, output of Alg. 2 is sensitive to (δ, η).

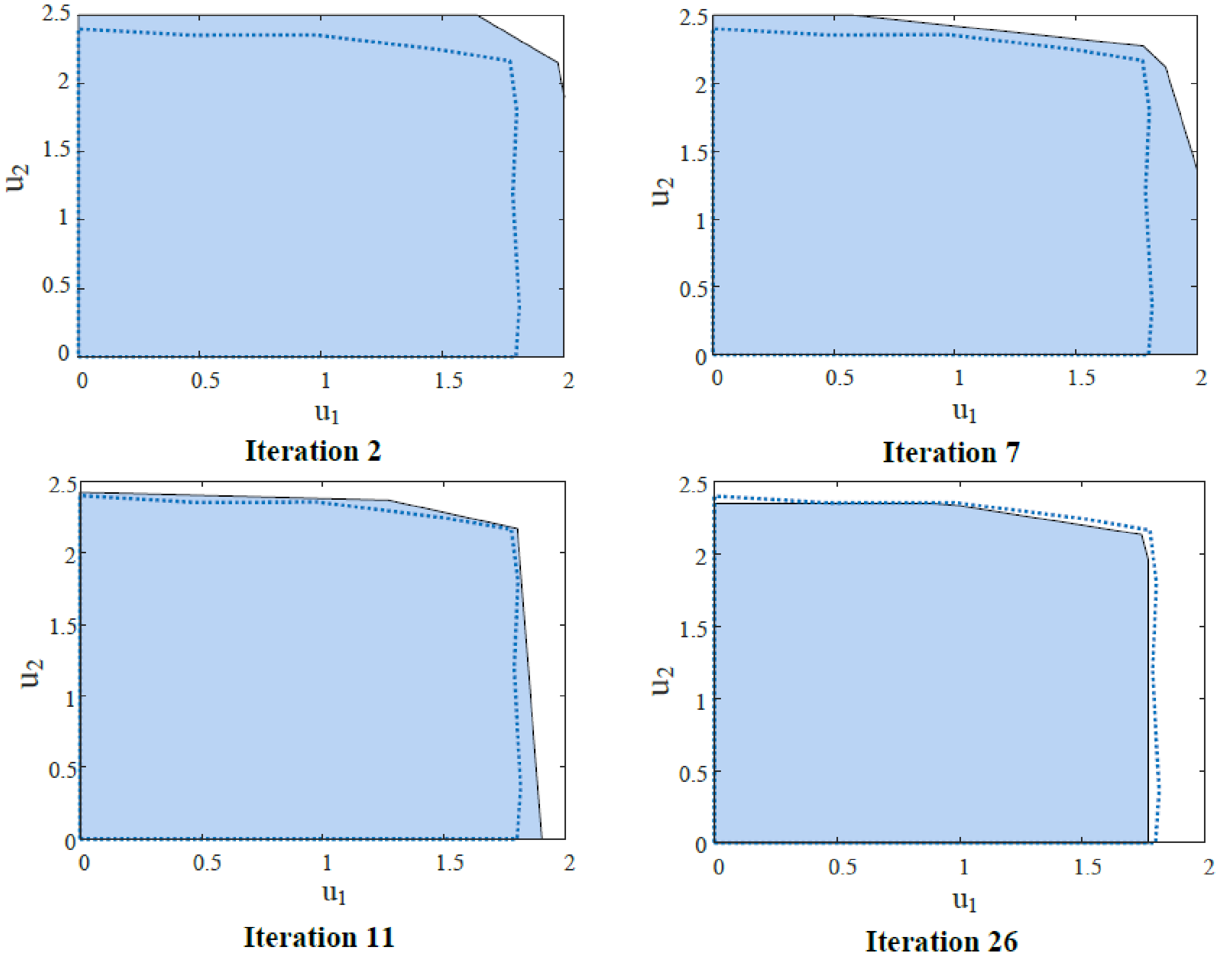
Proposed heuristic:

- Run Alg. 2 multiple times with different (δ, η)
- In each run, δ is a nonnegative perturbation to the dual optimal at one vertex of U'_{poly} (from Alg. 1)
- The union of multiple Alg. 2 outcomes serves as an approximation of $\widetilde{U}$

Preliminary numerical results

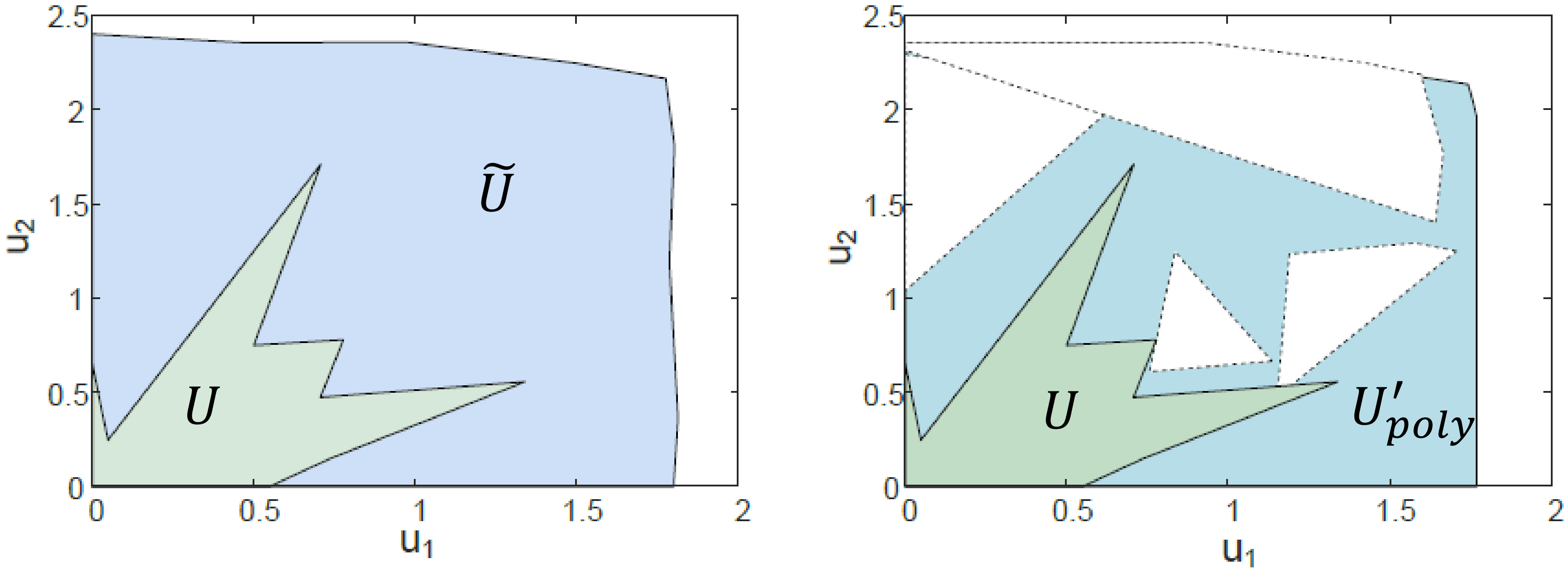






Alg. 1 converges in 26 iterations (263 seconds) to the SOCP-relaxed feasible region $\mathcal{U}'$

Preliminary numerical results



Left: Feasible region U and its SOCP relaxation $U' = U \cup \widetilde{U}$ found by checking sample points (close to *actual cases*)

Right: Alg. 1 output removing multiple Alg. 2 outputs.

Summary

A closed-form polyhedral approximation of feasible power injection regions in radial AC networks

- Model: nonlinear dist-flow
- Feasibility problem → SOCP relaxation → Dual SOCP → relaxed feasible region (Alg. 1)
- Heuristic to remove SOCP-inexact power injections (Alg. 2)
- Preliminary numerical results: simple form, moderate computation, relatively accurate

Limitations and future work: A better-justified design (rather than empirical heuristic) to remove SOCP-inexact power injections